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FILTRATION OF A LIQUID WITH FREE BOUNDARIES IN UNBOUNDED REGIONS

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The variational method is used to solve problems of filtration of a liquid in unbounded regions (inflow of a liquid to a drain, filtration of a liquid through a plain earth dam on a permeable base, etc.).

In the theory of steady flows of an incompressible liquid and gas, an important role belongs to variational principles developed by M. A. Lavrent'ev (1936) for conformal and quasi-conformal mapping. Another approach to these problems, which also has a variational character, was proposed in the papers of A. Weinstein (1924) and in the joint paper of J. Leray and A. Weinstein (1934). In contrast to these works, the variational method proposed by M. A. Lavrent'ev allowed him to establish not only theorems of existence of planar jet flows of a liquid but also theorems of uniqueness of the solutions under the same assumptions on the shape of obstacles. The method turned out to be also applicable to axisymmetric jets (J. Serrin, 1954).

In 1959, methods of M. A. Lavrent'ev, A. Weinstein, and J. Leray were further developed in the papers of V. N. Monakhov; the solvability of a wide class of planar steady problems of hydrodynamics with free boundaries was proved. As applied to the filtration theory, V. N. Monakhov proposed a variational method for proving the solvability of functional equations relative to the sought parameters of conformal mappings of finite regions with a polygonal shape of the specified part of the boundary. In the present paper, this method is extended to problems of liquid filtration in unbounded regions.

1. Formulation of the Problem. We study planar steady flows of an incompressible liquid in a porous medium (seam) with free (unknown) boundaries, which correspond to various hydrodynamic schemes of liquid filtration in the seam: inflow of a liquid to a drain or a well from a porous layer, liquid filtration from an open reservoir through a porous layer (for example, a plain earth dam or a porous insert in a chemical reactor), and liquid filtration under a hydrotechnological building whose underground part is found from given fields of pressures or velocities.

The case of an infinite depth of a saturated porous layer (filtration region of the half-plane type) is considered in a similar manner to the case of a finite region of filtration [1]. Therefore, we confine ourselves to the following two types of hydrodynamic filtration schemes: a liquid flow in a porous layer in the form of a half-band with one infinite apex and in a layer in the form of a band with two infinite apices [1–3].

We direct the Ox axis vertically upward, opposite to the vector of acceleration of gravity and perpendicular to the main direction of the filtration flow, and consider, in the plane of the complex variable $z = x + iy$, the filtration domain D bounded by the free boundary L (streamline), adjoining porous walls of the seam (equipotentials) P^1 (for $y > 0$) and P^2 (for $y < 0$), and the impermeable foot of the seam P^0 (streamline).

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The specified sectors $P^k \subset \partial D$ $(k = 1, 2)$ of the boundary ∂D are polygons with apices and ends at the points z_j^k $(j = \overline{1, n_k})$ and angles $\alpha_j^k \pi$ in them; the foot of the seam P^0 is assumed to be a straight line for simplicity.

We denote the point of intersection of P^1 and L as $z_1 \in P^1 \cap L$, $z_2 = P^1 \cap P^0 = \infty$, $z_3 = P^0 \cap P^2$ (possibly, $z_3 = \infty$), and $z_4 = P^2 \cap L$. In the vicinity of the point z_2 (and the point z_3, if $z_3 = \infty$), the polygon $(P^1 \cup P^0)$ (correspondingly, $P^0 \cup P^2$) is a half-band of width H_2 (H_3):

$$H_2 = \operatorname{Re}(z^1 - z^0) > 0 \quad \text{for} \quad z^k \in P^k, \quad \text{if} \quad \operatorname{Im} z^k \gg 1 \quad (k = 0, 1)$$

[correspondingly, we have $H_3 = \operatorname{Re}(z^2 - z^0) > 0$, where $z^k \in P^k$ and $k = 0$ and 2]. Similarly to the case of liquid filtration in a plane earth dam [1, p. 268], we set the quantity H $[H = \operatorname{Re}(z_1 - z_4) = |\operatorname{Re} z_4| > 0]$ of the acting (normalized) head of the liquid in the porous layer.

In the domain D, we seek an analytical function $w(z) = \varphi + i\psi$ (a complex potential of filtration), which satisfies the following boundary conditions on ∂D: $\varphi = \text{const}$ for $z \in P^1, P^2$, $\psi = \text{const}$ for $z \in P^0$, and $\varphi + x = \text{const}$ and $\psi = \text{const}$ for $z \in L$. In the plane w, the domain D corresponds to the rectangle $D^* = w(D)$ with apices at the points w_k $(k = \overline{1, 4})$, which are operands of z_k, $|w_1 - w_4| = |w_2 - w_3| = H$ is a given liquid head and $|w_1 - w_2| = |w_3 - w_4| = Q$ is the sought flow rate. Note that, in the case considered, the region of leaking (drainage) [1–3] is horizontal. The derivatives of conformal mappings of the upper half-plane $E = \{\operatorname{Im} \zeta > 0\}$ onto the domains D and D^* have the following form [1]:

$$\frac{dw}{d\zeta} = \prod_1^4 (\tau_k - \zeta)^{-1/2} = \Pi_0(\zeta), \qquad \frac{dz}{d\zeta} = \Pi(\zeta) M(\zeta),$$
$$\Pi = \prod_{k,j} (\zeta - t_j^k)^{\beta_j^k} (\zeta - \tau_2)^{-1} (\zeta - \tau_3)^{-\delta}, \qquad M = \frac{1}{\pi i} \int_{-1}^{1} \frac{|\Pi_0(t)|}{|\Pi(t)|(t - \zeta)}\, dt. \tag{1}$$

Here t_j^k are the operands of the apices z_j^k $(k = 1, 2;\ j = \overline{1, n_k})$ of the polygon $(P^1 \cup P^2)$, τ_k are the operands of the points z_k $(k = \overline{1, 4})$, and $\beta_j^k \pi = (\alpha_j^k - 1)\pi$ are the external angles of the polygon $(P^1 \cup P^2)$; $\delta = 0$ for $|z_3| < \infty$ and $\delta = 1$ for $z_3 = \infty$. We normalize the conformal mapping $z(\zeta)$, $z : E \to D$ assuming that $\tau_4 = t_{n_2}^2 = -1$, $\tau_1 = t_1^1 = 1$, $t_{n_2-1}^2 = -2$, $z_1 = z(\tau_1) = H + H_2$, $\tau_1 \leqslant t_j^1 < t_{j+1}^1 < \tau_2$ $(j = \overline{1, n_1 - 1})$, and $\tau_3 \leqslant t_j^2 < t_{j+1}^2 \leqslant \tau_4$ $(j = \overline{1, n_2 - 1})$.

The unknown constants τ_2, τ_3, and t_j^k $(k = 1,\ j = \overline{2, n_1};\ k = 2,\ j = \overline{2, n_2 - 2})$ are found from the following system of equations, which defines the geometry of the polygon $P = \bigcup_{k=0}^{2} P^k$:

$$l_j^k = \int_{t_j^k}^{t_{j+1}^k} \left| \frac{dz}{dt} \right| dt \qquad (k = 1, \quad j = \overline{1, n_1 - 1}; \quad k = 2, \quad j = \overline{1 + \gamma, n_2 - 2}),$$
$$H_i = \pi M(\tau_i) \Pi_i(\tau_i), \qquad i = 2, 2 + \gamma \qquad (n_1 \geqslant 1, \quad n_2 \geqslant 3). \tag{2}$$

Here $\Pi_i = \Pi(\zeta)(\zeta - \tau_i)$ $(i = 2, 3)$; $\gamma = 0$ for $|z_3| < \infty$ and $\gamma = 1$ for $z_3 = \infty$.

Note that, according to conditions (2), not all lengths of the segments of the polygon P^2 are fixed. This is related to the presence of a horizontal section of leaking (drainage) whose length is a sought quantity in the vicinity of the point $z_4 = P^2 \cap L$.

2. *A Priori* Estimates. We use the notation $\alpha = (\alpha_1^1, \ldots, \alpha_{n_1}^1; \alpha_1^2, \ldots, \alpha_{n_2}^2) \in R^n (n = n_1 + n_2)$ for the *characteristic of interior angles* $\alpha_j^k \pi$ of the polygon $P = \bigcup_0^2 P^k$, $l = (l_1^1, \ldots, l_{n_1}^1; l_1^2, \ldots, l_{n_2-2}^2)$ $(l_{n_1}^1 \equiv H_2$, $l_1^2 \equiv H_3$ for $z = \infty)$ for the *metric characteristic* of P, and call $p = (\alpha, l)$ *the geometric characteristic* of P.

We consider the family $G(\delta)$ of simple polygons $P \subset G$ with the characteristics $p \in G(\delta)$:

$$G(\delta) = \begin{cases} 0 < \delta \leqslant \alpha_j^k \leqslant 2, \quad (j,k) \in I; \quad 0 \leqslant (\alpha_1^1, \alpha_{n_2}^2) \leqslant 3/2 - \delta, \\ \sum\limits_{j=1}^{n_k} (\alpha_j^k - 1) = 0, \quad k = 1, 2 \quad (\alpha_1^2 = 0 \quad \text{for} \quad z_3 = \infty), \\ |\ln l_j^k| \leqslant \delta^{-1}, \quad j = \overline{1, n_1}, \quad k = 1; \quad j = \overline{1, n_2 - 2}, \quad k = 2, \end{cases} \tag{3}$$

where $I = (j = \overline{2, n_1}, k = 1; j = \overline{1, n_2 - 1}, k = 2)$. The condition $\sum\limits_{j=1}^{n_k} (\alpha_j^k - 1) = 0$ ensures the validity of the necessary estimate $0 < |\zeta|^2 |z_\zeta| < \infty$ in the vicinity of $\zeta = \infty$.

We assume that $\Delta t_j^k = |t_{j+1}^k - t_j^k|$ $(j = \overline{1, n_k - 1}, k = 1, n - 2)$ and consider $u = (t_1^1, \dots, t_{n_1}^1; t_1^2, \dots, t_{n_2-2}^2) \in R^{n-2}$ $(t_{n_1+1}^1 \equiv \tau_2, t_1^2 \equiv \tau_3)$.

For the solution $u \in R^{n-2}$ of system (2) corresponding to the simple polygon $P \subset G(\delta)$, we establish the validity of the following inclusion (*a priori* estimates):

$$u \in \Omega = \{u \mid 0 < \varepsilon(\delta) \leqslant \Delta t_j^k \leqslant \varepsilon^{-1}, \quad j = \overline{1, n_k - 1}, \ k = 1, 2\}. \tag{4}$$

Here the constant $\varepsilon(\delta) > 0$ depends only on the geometric characteristic p of the polygon P in (3).

We consider one corollary of system (2):

$$H = \int_{-1}^{1} \Pi_0(t)\, dt, \qquad \Pi_0 = \prod_{k=1}^{4} |t - \tau_k|^{-1/2}.$$

Taking into account that $|\tau_3| \leqslant |t_{n_2-1}^2| = 2$, we find $H \leqslant K_1(\tau_2 - 1)^{-1/2}$, whence we obtain $\tau_2 - 1 \leqslant (H^{-1}K_1)^2 \equiv K_2$.

Let $r \equiv (\tau_2 - 1) \to 0$. Then we have

$$H \geqslant \int_0^1 \Pi_0\, dt \geqslant K_3^{-1} \int_0^1 (1 + r - t)^{-1}\, dt, \quad \text{i.e.} \quad r \geqslant (e^{HK_3} - 1)^{-1} \equiv \varepsilon_2 > 0.$$

Coming back to the relation for H, we obtain $H \leqslant K_4 \varepsilon_2^{-1/2} (|\tau_3| - 1)^{-1/2}$, whence we have $|\tau_3| \leqslant K_5$.

We now establish the validity of the estimate $t_{n_1}^1 - 1 \geqslant \varepsilon > 0$. Assume, to the contrary, that $r \equiv (t_{n_1}^1 - 1) \to 0$. We introduce the auxiliary function

$$M_r(\zeta) = \frac{1}{\pi i} \int_{-1}^{1-r} \frac{|\Pi_0(t)\Pi^{-1}(t)|\, dt}{t - \zeta}, \quad M_r(\zeta) \to M(\zeta), \quad r \to 0, \quad |\zeta| < \infty.$$

We consider the half-circle $K_r = \{|\zeta - 1 - r/2| = r\} \cap \{\operatorname{Im} \zeta > 0\}$ and show that

$$|\Lambda_r| = \left| \int_{K_r} \Pi(\zeta) M_r(\zeta)\, d\zeta \right| \to 0 \quad \text{as} \quad r \to 0.$$

In this case, obviously, we have $l_k^1(r) \to 0$ $(k = \overline{1, n_1 - 1})$, which is impossible; thus, we obtain $t_{n_1} - 1 \geqslant \varepsilon > 0$.

In accordance with the geometry of the polygon $P = \bigcup\limits_0^2 P^k$, we have $\sum\limits_{k=1}^{n_1} \beta_k^1 = 0$. We assume that $\sum\limits_1^{n_1} = \Sigma' + \Sigma''$, with all $\beta_k^1 \leqslant 0$ collected in Σ' and all $\beta_k^1 > 0$ in Σ'', and note that $\Sigma'' \beta_k^1 = -\Sigma' \beta_k^1 \equiv \mu > 0$. Taking this into account, we find

$$|M_r(\zeta)| \leqslant C \int_{-1}^{1-r} (1 + t)^{1/2 - \alpha_{n_2}^2} \varphi(t) \frac{(1 - t)^{-1/2}}{|t - \zeta|}\, dt, \quad \varphi = \left(1 + \frac{r}{1 - t}\right)^{\mu}.$$

Since $\varphi(t) \leqslant 2^\mu$ for $t \in [-1, 1-r]$, then we have

$$|M_r(\zeta)|\,|\zeta - 1|^{1/2} \leqslant C_0 \qquad (\zeta \in K_r, \qquad r \leqslant 1/2),$$

which leads to the estimate $|\Lambda_r| \leqslant C_1 r^{1/2}$. In this case, obviously, we obtain $l_k^1(r) \to 0$ ($k = \overline{1, n_1 - 1}$), which is impossible, i.e., $t_n^1 - 1 \geqslant \varepsilon > 0$. It is proved in a similar manner that $t_k^1 - 1 \geqslant \varepsilon > 0$ for $k = \overline{2, n_1 - 1}$.

We assume that $\tau_2 - t_p^1 \equiv r \to 0$ ($p = \overline{2, n_1}$). If $\beta^p \equiv \sum_{k=p}^{n_1} \beta_k^1 \neq 0$, then in relation (2) we obtain $H_2 = \pi|\Pi_2(\tau_2)M(\tau_2)|$, $M(\tau_2) \neq 0, \infty$, and $\Pi_2(\tau_2) \to 0$ for $\beta^p > 0$ and $\Pi_2(\tau_2) \to \infty$ for $\beta^p < 0$, which contradicts condition (3); therefore, we have $|\ln H_2| \leqslant \delta^{-1} < \infty$. Let $\beta^p = \sum_{k=p}^{n_1} \beta_k^1 = 0$. We assume that $\sum_{k=p}^{n_1} \beta_k^1 \equiv \Sigma' + \Sigma''$ with all $\beta_k^1 \geqslant 0$ collected in Σ' and all $\beta_k^1 < 0$ in Σ''. We use the notation $\Sigma'\beta_k^1 = \mu > 0$ and $\Sigma''\beta_k^1 = -\nu$; in accordance with the assumption $\beta^p = 0$, we have $\mu - \nu = 0$. We consider the expression for l_{p-1} ($p \geqslant 2$) in (2), taking into account that t_{p-1}^1 does not belong to converging parameters; therefore, we have $t_p^1 - t_{p-1}^1 \geqslant 2\varepsilon > 0$ (ε is fixed). We assume that $t_p^1 \equiv \tau$ and $\tau_2 - \tau \equiv r \to 0$. Then we obtain

$$l_{p-1} \geqslant \int_{\tau-\varepsilon}^{\tau} \left|\frac{dz}{dt}\right| dt \geqslant K \int_{\tau-\varepsilon}^{\tau} (\tau - t)^\mu (\tau + r - t)^{-\nu}\, dt \equiv K I(r).$$

We perform the substitution $\tau - t = sr$ in the integral $I(r)$:

$$I(r) = \int_0^{\varepsilon/r} s^\mu (1+s)^{-\nu}\, ds \geqslant \int_1^{\varepsilon/r} \left(1 + \frac{1}{s}\right)^{-\nu} ds \to \infty \quad \text{as} \quad r \to 0,$$

which contradicts condition (3) $|\ln l_{p-1}| \leqslant \delta^{-1} < \infty$. Thus, we have established

$$1 + \varepsilon_1 \leqslant t_k^1 \leqslant \tau_2 - \varepsilon_2, \qquad k = \overline{2, n_1}, \qquad (\varepsilon_1, \varepsilon_2) > 0.$$

These inequalities make it possible to use the estimates $u_k^1 = t_{k+1}^1 - t_k^1 \geqslant \varepsilon > 0$, where $k = \overline{1, n_1}$ ($t_{n_1+1} \equiv \tau_2$), which are valid in the case of a finite polygon $P = \bigcup_0^2 P^k$ [1]. Similar considerations are also applicable in proving the estimates $\Delta t_k^2 = t_{k+1}^2 - t_k^2 \geqslant \varepsilon > 0$, where $k = \overline{1, n_2 - 1}$ ($t_1^2 \equiv \tau_3$), corresponding to the polygon P^2. The *a priori* estimates (4) are proved.

Remark 1. The main difficulty in obtaining estimates (4) is the fact that the density $h(t) \equiv \prod_1^4 |t - \tau_4|^{-1/2}$ of the integral $M(\zeta)$ in (1) depends on the sought constants τ_2 and τ_3 ($\tau_1 = 1$ and $\tau_4 = -1$). In an appropriate representation of $M(\zeta)$ in [1, p. 111], $h(t)$ is a function of only prescribed constants τ_1 and τ_4.

Remark 2. As follows from the proof of the *a priori* estimates (4), even if $P^0 \in G(\delta)$ is not a straight line, the validity of (4) is obviously retained in this case too.

Remark 3. Another normalization of conformal mappings defined in (1) is possible: $\tau_1 = 1$, $\tau_4 = -1$, and $\tau_3 = -2$. Then from the relation

$$H(\tau_2) = \int_{-1}^{1} |\Pi_0(t)|\, dt \quad \left[\frac{dH}{d\tau_2} < 0,\ H(1) = \infty,\ H(\infty) = 0\right],$$

we can uniquely determine τ_2 and, hence, the flow rate of the liquid

$$Q = \int_1^{\tau_2} |\Pi_0(t)|\, dt.$$

In this case, one equation in system (1) should be rejected, for example, it is not allowed to set the value of H_2. Therefore, this normalization is unphysical.

3. Local Uniqueness of the Solutions. We write system (2) in an operator form with respect to $u = (t_1^1, \dots, t_{n_1}^1; t_1^2, \dots, t_{n_2-2}^2) \equiv (u_1, \dots, u_{n-2}) \in R^{n-2}$ $(n = n_1 + n_2)$:

$$l = g(u, \alpha) = (g_1, \dots, g_{n-2}), \tag{5}$$

where $l = (l_1^1, \dots, l_{n_1}^1; l_1^2, \dots, l_{n_2-1}^2) \equiv (l_1, \dots, l_{n-2})$ is the metric characteristic of P and α is the characteristic of interior angles of P.

We prove the following properties of the operator $g(u, \alpha)$:

$$g(u, \alpha) \in C^2(\Omega \times G); \qquad \frac{Dg}{Du} = \{g_{ij}\} \neq 0, \qquad g_{ij} = \frac{\partial g_i}{\partial u_j}, \quad u \in \Omega. \tag{6}$$

The sets Ω and G are defined in (3) and (4).

Differentiability of the components g_i representable in the form

$$g_i = l_j^k = \int\limits_{t_j^k}^{t_{j+1}^k} \left| \frac{dz}{dt} \right| dt,$$

follows from [1] after reducing the integration intervals to $[0, 1]$. For the components $g_{n_1} = H_2$ and $g_{n_1+1} = H_3$, $H_k = \pi |\Pi_k(\tau_k) M(\tau_k)|$ $(k = 2, 3)$, differentiability on the set $(u, p) \in (\Omega \times G)$ can be verified directly.

We prove the nondegeneracy of the transformation $l = g(u, \alpha)$, $Dg/Du \neq 0$ by a variational method in a similar manner to [1]. We express the variation of l for a fixed α via the variation of the sought solution u in (5): $\delta l = (Dg/Du)\,\delta u$. Assuming that $\delta u \neq 0$ for $\delta l = 0$, in the resultant equality we calculate the variations of the mappings $z : E \to D$ and $\zeta : D \to E$ through each other: $\delta z + z_\zeta \delta\zeta = 0$. Posing a boundary-value problem for δz from this relation, we obtain $\delta z = \Pi(\zeta) Q_{m_0}(\zeta)$, where $Q_{m_0}(\zeta)$ is a polynomial of power $m_0 \geqslant 0$. We now calculate δz directly from the representation $z = z(\zeta)$:

$$z = \int\limits_1^\zeta \Pi(\zeta) M(\zeta)\, d\zeta + z_1, \qquad \delta z = \int\limits_1^\zeta \Pi(\zeta) \Phi(\zeta, \delta u)\, d\zeta \qquad (\delta z_1 = 0),$$

$$\Phi = \sum_{j,k} \left[(1 - \alpha_j^k)(\zeta - t_j^k)^{-1} M(\zeta) + \frac{\partial M}{\partial t_j^k} \right] \delta t_j^k.$$

Note that $|\delta z(\infty)| < \infty$. Comparing δz and $(\delta z)_\zeta$ in the vicinity of t_j^k, from the resultant representation with δz and $(\delta z)_\zeta$ found by solving the boundary-value problem, we finally obtain

$$\delta z = \prod_{j,k} (\zeta - t_j^k)^{\alpha_j^k - \gamma_j^k} Q_m(\zeta) (\zeta - \tau_2)^{-1} (\zeta - \tau_3)^{-\delta}. \tag{7}$$

Here $\gamma_j^k = 0$ if $\delta t_j^k = 0$ and $\gamma_j^k = 1$ for $\delta t_j^k \neq 0$; $\delta = 0$ for $|z_3| < \infty$ and $\delta = 1$ for $z_3 = \infty$; $Q_m(\zeta)$ is a polynomial of power m. In this case, we have $\lambda \equiv \sum_{j,k} \alpha_j^k = n_1 + n_2 = n$, for $|z_3| < \infty$ and $\lambda = 1$ for $z_3 = \infty$. Since $\delta t_1^1 = \delta t_{n_2}^2 = \delta t_{n_2-1}^2 = 0$, we obtain $\sum_{j,k} \gamma_j^k \leqslant n - 3$. Then, according to representation (7), in the vicinity of $\zeta = \infty$ we have $|\delta z|\,|\zeta|^{-q} < \infty$, where $q = \sum_{j,k} (\alpha_j^k - \gamma_j^k) + m_0 - 1 - \delta \geqslant 2$, which contradicts the boundedness of $\delta z(\infty)$. Thus, we have $\delta z \equiv 0$, whence it necessarily follows that $\Phi(\zeta, \delta u) \equiv 0$ in the representation for δz, which, in turn, involves the equality $\delta u = 0$. Relations (6) are proved.

4. Existence and Uniqueness of the Solutions. The *a priori* estimates (4) and the local uniqueness (6) of the solutions of system (2) corresponding to a simple polygon $P \subset G(\delta)$ defined in (3) allow us to use the method of continuity to prove its solvability [1]. The variant of the method of continuity developed in [1] involves the construction of local variations of the initial polygon P_0 for which the solvability of (2) is known, by transforming P_0 to a given polygon P. By virtue of (6), we have $Dg/Du \neq 0$, and the solvability of (2) for a small deformation of P_0 follows from the theorem of implicit functions.

An algorithm for constructing a family of polygons P_k converging to a given polygon P is proposed in [1], and the solvability of (2) is proved on the grounds of its solvability for the initial polygon P_0.

In jet problems of hydrodynamics [1, Chapter 4], this algorithm is also implemented in the case of infinite regions. This generalization is transferred in a similar manner to problems of filtration theory for which properties (4) and (6) are established. The theorem of uniqueness of the solutions of system (2) for a given polygon $P \subset G(\delta)$ also follows from the method of continuity if this theorem is valid for the initial polygon P_0.

Let us construct first an initial polygon P_0 for problems of filtration theory in domains of the type of a half-band ($|z_3| < \infty$). We assume that $P_0^0 = \{x = 0,\ y > y_3 = \operatorname{Im} z_3\}$, $P_0^1 = \{x = H_2 > H, y > 0\}$, and $P_0^2 = \{0 < x < H_2 - H, y = x\sin(1-\alpha)\}$, where $\alpha \in (1/2, 1)$ [condition (3)]. The head H is given, and the depth H_2 is not fixed yet. Then in (1) we have

$$\Pi(\zeta) = (\zeta - \tau_2)^{-1}(\zeta - \tau_3)^{\alpha-1}(\zeta - \tau_4)^{1-\alpha}, \qquad \tau_1 = 1, \qquad \tau_4 = -1.$$

In addition, we fix the constant $\tau_3 = -2$ and from the condition

$$H = \int_{-1}^{1} |\Pi_0(t)|\, dt \equiv H(\tau_2) \qquad \Big(\frac{dH}{d\tau_2} < 0, \quad H(\infty) = 0, \quad H(1) = \infty\Big)$$

we uniquely determine τ_2 and, consequently, $H_2 = \pi|\Pi_2(\tau_2)M(\tau_2)|$ in (1).

If there are apices z_j^k ($j = \overline{1, n_k}$, where $k = 1$ and 2) with angles $\alpha_j^k\pi$ at them on P^k in the initial polygon $P = \bigcup_0^2 P^k$, then for the polygon P_0 we introduce fixed operands t_{0j}^k of the "apices" $z_{0j}^k \in P_0^k$ ($z_1^1 = z_1$, $z_1^2 = z_3$, and $z_{n_2}^2 = z_4$) with angles $\alpha_{0j}^k\pi = \pi$. They should obey the conditions

$$t_{01}^1 = 1 < t_{0j}^1 < t_{0j+1}^1 < \tau_2, \qquad t_{01}^2 = \tau_3 < t_{0j}^2 < t_{0j+1}^2 < \tau_4.$$

Using the constructed conformal mapping $z = z_0(\zeta)$, $z_0\colon E \to D_0, P_0 \subset \partial D_0$, we uniquely determine the apices $z_{0j}^k = z_0(t_{0j}^k)$ and, consequently, $l_{0j} = |z_{0j+1}^k - z_{0j}^k|$.

System (2) corresponding to the thus-fixed polygon $P_0 = \bigcup_1^2 P_0^k$ ($z_{0j}^k \subset P_0^k$, $\alpha_{0j}^k = 1$) in terms of construction is uniquely solvable with respect to $u_0 = (t_{02}^1, \dots, t_{0n_1}^1, \tau_2; t_1^2, \dots, t_{n_1-2}^2)$, i.e., P_0 has the required properties of the initial polygon. The deformation of P_0 to the prescribed polygon P corresponding to the initial problem of filtration theory is now constructed using a standard procedure [1, Chapters 3 and 4].

Let $z_3 = \infty$. We construct a polygon $P_0 = \bigcup_0^2 P_0^k$:

$$P_0^0 = \{x = 0, -\infty < y < \infty\}, \quad P_0^1 = \{x = H_2 > H,\ y > 0\},$$

$$P_0^2 = \{x = H_2 - H;\ y < y_0 = \operatorname{Im} z_0,\ y_0 < y < y_4 = \operatorname{Im} z_4\},$$

and there is an angle $\alpha_0\pi = 2\pi$ at the point $z_0 \in P_0^2$ (section of P_0^2). The head H and the depth H_2 are set. In (1), we have

$$\Pi(\zeta) = [(\zeta - \tau_2)(\zeta - \tau_3)]^{-1}(\zeta - \tau_0), \quad z_0 = z_0(\tau_0), \quad z_0 : E \to D_0, \quad P_0 \subset \partial D_0.$$

As in the previous case, we fix $\tau_1 = 1$, $\tau_4 = -1$, and $\tau_3 = -2$, thus, defining τ_2 from the condition $H = \int_{-1}^{1} |\Pi_0(t)|\, dt$.

We now consider the equation for $H_3 = H_2 - H > 0$:

$$H_3(\tau_0) = \pi \frac{\tau_0 - \tau_3}{\tau_2 - \tau_3}|M(\tau_3)| \qquad \left(\frac{dH_3}{d\tau_0} > 0, \quad H_3(\tau_3) = 0, \quad H_3(\tau_4) = \infty\right),$$

from which we uniquely find $\tau_0 \in (\tau_3, \tau_4)$.

If necessary, we fix the constants t_{0j}^k and construct the points $z_{0j}^k = z_0(t_{0j}^k)$, $z_0 : E \to D_0, P_0 \subset \partial D_0$.

System (2) corresponding to the constructed polygon P_0 is, obviously, uniquely solvable; hence, P_0 has all the necessary properties of the initial polygon. We prove the following theorem.

Theorem 1. *Let liquid filtration occur in a domain D bounded by a free boundary L and a simple polygon $P = \bigcup_0^2 P^k \subset G$ [condition (3)]. Then system (2) with respect to the vector $u \in R^{n-2}$ of the sought parameters of the conformal mapping $z : E \to D$, $\partial D = P \cup L$ and, hence, the initial problem of filtration theory are uniquely solvable.*

Remark 4. In [1, Chapters 3 and 4], the family $P_m^k \to \Gamma^k$ $(m \to \infty)$ is used to justify the limiting transition to given curved boundaries Γ^k $(k = 1, 2)$, which is also applicable in the examined problems of the filtration theory. However, the uniqueness of the solutions is not guaranteed in the limiting case.

Remark 5. For curved boundaries $\Gamma^k \subset \partial D$, the theorem of existence and uniqueness of filtration problems can be established by other methods [1, Chapter 8, § 5].

Let, for definiteness, $z_3 = \infty$,

$$\Gamma^k: \quad x = f^k(y), \quad |y| \geqslant y_0^k \quad (y_0^1 = 0, \quad y_0^2 = \operatorname{Im} z_0).$$

We assume that $f^k(y) \in C^2(\Gamma^k)$, $f^k \equiv x^k = \text{const}$ for $|y| \geqslant y_1^k > y_0^k$, and $df^k/dy \neq 0$ for $y_0^k < |y| < y_1^k$. We make the substitution of variables: $x - f^k(y) = \xi - x^k$ and $y = \eta$. Then Γ^k in the new variables will be transformed to straight half-lines P^k (second example of P_0). The resultant simplest boundary-value problem for the generalized analytical function $z = F(\zeta)$ has a unique solution [1, p. 388].

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